BER for BPSK in Rayleigh channel

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Long back in time we discussed the [BER (bit error rate) for BPSK modulation](http://www.dsplog.com/2007/08/05/bit-error-probability-for-bpsk-modulation/) in a simple AWGN channel (time stamps states August 2007). Almost an year back! It high time we discuss the **BER for BPSK** in a **Rayleigh multipath channel**.

In a brief [discussion on Rayleigh channel](http://www.dsplog.com/2008/07/14/rayleigh-multipath-channel/), wherein we stated that a circularly symmetric complex Gaussian random variable is of the form,

http://www.dsplog.com/cgi-bin/mimetex.cgi?h%20=%20h_%7bre%7d%20+%20jh_%7bim%7d,

where real and imaginary parts are zero mean independent and identically distributed (iid) Gaussian random variables with mean 0 and variance http://www.dsplog.com/cgi-bin/mimetex.cgi?\sigma%5e2.

The magnitude http://www.dsplog.com/cgi-bin/mimetex.cgi?|h| which has a probability density,

**http://www.dsplog.com/cgi-bin/mimetex.cgi?p%28h%29%20=%20%5Cfrac%7Bh%7D%7B%5Csigma%5E2%7De%5E%7B%5Cfrac%7B-h%5E2%7D%7B2%20%5Csigma%5E2%7D%7D,%5C%20%5C%20%5C%20%20%20%20%20%20z%5Cge%200**

is called a**Rayleigh random variable.**This model, called **Rayleigh fading channel model**, is reasonable for an environment where there are large number of reflectors.

**System model**

The received signal in **Rayleigh fading channel** is of the form,

http://www.dsplog.com/cgi-bin/mimetex.cgi?y=hx+n, where  
http://www.dsplog.com/cgi-bin/mimetex.cgi?yis the received symbol,  
http://www.dsplog.com/cgi-bin/mimetex.cgi?his complex scaling factor corresponding to **Rayleigh** multipath channel  
http://www.dsplog.com/cgi-bin/mimetex.cgi?x is the transmitted symbol (taking values +1′s and -1′s) and  
http://www.dsplog.com/cgi-bin/mimetex.cgi?n is the Additive White Gaussian Noise (AWGN)

**Assumptions**

1. The channel is flat fading – In simple terms, it means that the multipath channel has only one tap. So, the convolution operation reduces to a simple multiplication. For a more rigorous discussion on flat fading and frequency selective fading, may I urge you to review Chapter 15.3 **Signal Time-Spreading** from [[DIGITAL COMMUNICATIONS: SKLAR]](http://www.amazon.com/gp/redirect.html?ie=UTF8&location=http%3A%2F%2Fwww.amazon.com%2FDigital-Communications-Fundamentals-Applications-Technologies%2Fdp%2F0130847887&tag=dl04-20&linkCode=ur2&camp=1789&creative=9325)

2. The channel is randomly varying in time – meaning each transmitted symbol gets multiplied by a randomly varying complex number http://www.dsplog.com/cgi-bin/mimetex.cgi?h. Since http://www.dsplog.com/cgi-bin/mimetex.cgi?his modeling a Rayleigh channel, the real and imaginary parts are Gaussian distributed having **mean 0** and **variance 1/2**.

3. The noisehttp://www.dsplog.com/cgi-bin/mimetex.cgi?n has the Gaussian probability density function with

http://www.dsplog.com/cgi-bin/mimetex.cgi?p%28n%29%20=%20%5Cfrac%7B1%7D%7B%5Csqrt%7B2%5Cpi%5Csigma%5E2%7D%7De%5E%7B%5Cfrac%7B-%28n-%5Cmu%29%5E2%7D%7B2%5Csigma%5E2%7D with http://www.dsplog.com/cgi-bin/mimetex.cgi?%5Cmu=0 and http://www.dsplog.com/cgi-bin/mimetex.cgi?%5Csigma%5E2%20=%20%5Cfrac%7BN_0%7D%7B2%7D.

4. The channel http://www.dsplog.com/cgi-bin/mimetex.cgi?his known at the receiver. Equalization is performed at the receiver by dividing the received symbol http://www.dsplog.com/cgi-bin/mimetex.cgi?yby the apriori known http://www.dsplog.com/cgi-bin/mimetex.cgi?h i.e.

http://www.dsplog.com/cgi-bin/mimetex.cgi?\hat%7by%7d%20=%20\frac%7by%7d%7bh%7d%20=%20\frac%7bhx+n%7d%7bh%7d%20=%20x+\tilde%7bn%7d

where http://www.dsplog.com/cgi-bin/mimetex.cgi?\tilde%7bn%7d%20=%20\frac%7bn%7d%7bh%7dis the additive noise scaled by the channel coefficient.

**Bit Error Rate**

The equations listed below refers Chapter 14.3 in [[DIGITAL COMMUNICATION: PROAKIS]](http://www.amazon.com/gp/redirect.html?ie=UTF8&location=http%3A%2F%2Fwww.amazon.com%2FDigital-Communications-John-Proakis%2Fdp%2F0072321113&tag=dl04-20&linkCode=ur2&camp=1789&creative=9325)

If you recall, in the post on**BER computation in AWGN**, the probability of error for transmission of either +1 or -1 is computed by integrating the tail of the Gaussian probability density function for a given value of bit energy to noise ratio http://www.dsplog.com/cgi-bin/mimetex.cgi?\frac%7bE_b%7d%7bN_0%7d. The bit error rate is,

http://www.dsplog.com/cgi-bin/mimetex.cgi?P_b=%5Cfrac%7B1%7D%7B2%7Derfc%5Cleft%28%7B%5Csqrt%7B%5Cfrac%7BE_b%7D%7BN_0%7D%7D%7D%5Cright%29.

However in the presence of channel http://www.dsplog.com/cgi-bin/mimetex.cgi?h, the effective bit energy to noise ratio is http://www.dsplog.com/cgi-bin/mimetex.cgi?\frac%7b|h|%5e2E_b%7d%7bN_0%7d. So the bit error probability for a given value of http://www.dsplog.com/cgi-bin/mimetex.cgi?his,

http://www.dsplog.com/cgi-bin/mimetex.cgi?P_%7bb|h%7d=%5Cfrac%7B1%7D%7B2%7Derfc%5Cleft%28%7B%5Csqrt%7B%5Cfrac%7B|h|%5e2E_b%7D%7BN_0%7D%7D%7D%5Cright%29=\frac%7b1%7d%7b2%7derfc\left(\sqrt%7b\gamma%7d\right), where http://www.dsplog.com/cgi-bin/mimetex.cgi?\gamma%20=%20\frac%7b|h|%5e2E_b%7d%7bN_0%7d.

To find the error probability over all random values of http://www.dsplog.com/cgi-bin/mimetex.cgi?|h|%5e2, one must evaluate the conditional probability density function http://www.dsplog.com/cgi-bin/mimetex.cgi?P_%7bb|h%7d over the probability density function of http://www.dsplog.com/cgi-bin/mimetex.cgi?\gamma.

**Probability density function of http://www.dsplog.com/cgi-bin/mimetex.cgi?\gamma**

From our discussion on [chi-square random variable](http://www.dsplog.com/2008/07/28/chi-square-random-variable/#twodegree), we know that if http://www.dsplog.com/cgi-bin/mimetex.cgi?|h|is a Rayleigh distributed random variable, then http://www.dsplog.com/cgi-bin/mimetex.cgi?|h|%5e2 is chi-square distributed with two degrees of freedom. since http://www.dsplog.com/cgi-bin/mimetex.cgi?|h|%5e2 is chi square distributed, http://www.dsplog.com/cgi-bin/mimetex.cgi?\gamma is also chi square distributed. The probability density function of http://www.dsplog.com/cgi-bin/mimetex.cgi?\gammais,

http://www.dsplog.com/cgi-bin/mimetex.cgi?p\left(\gamma\right)%20=%20\frac%7b1%7d%7b(E_b/N_0)%7de%5e%7b\frac%7b-\gamma%7d%7b(E_b/N_0)%7d%7d,\%20\gamma%20\ge%200.

**Error probability**

So the error probability is,

http://www.dsplog.com/cgi-bin/mimetex.cgi?P_%7bb%7d=\int_0%5e\infty\frac%7b1%7d%7b2%7derfc\left(\sqrt%7b\gamma%7d\right)p\left(\gamma\right)d\gamma.

Somehow, this equation reduces to

http://www.dsplog.com/cgi-bin/mimetex.cgi?\Large%20P_%7bb%7d=\frac%7b1%7d%7b2%7d\left(1-\sqrt%7b\frac%7b(E_b/N_0)%7d%7b(E_b/N_0)%20+1%7d%7d\right).

Note:

*1. I have not yet figured out the math to reduce the above integral to the answer. If some one knows, kindly drop in a comment.*

*2. Another way for finding the bit error rate might be to find the pdf of http://www.dsplog.com/cgi-bin/mimetex.cgi?\tilde%7bn%7d. However, I do not know how to find pdf following the division of two random variables. :)*

**Simulation Model**

It will be useful to provide a simple Matlab/Octave example simulating a BPSK transmission and reception in Rayleigh channel. The script performs the following

(a) Generate random binary sequence of +1′s and -1′s.

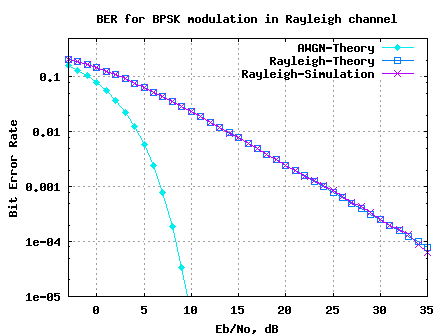
(b) Multiply the symbols with the channel and then add white Gaussian noise.

(c) At the receiver, equalize (divide) the received symbols with the known channel

(d) Perform hard decision decoding and count the bit errors

(e) Repeat for multiple values of http://www.dsplog.com/cgi-bin/mimetex.cgi?\frac%7bE_b%7d%7bN_0%7d and plot the simulation and theoretical results.

Click here to download [Matlab/Octave script for BER computation of BPSK in Rayleigh fading channel](http://www.dsplog.com/db-install/wp-content/uploads/2008/08/script_ber_bpsk_rayleigh_channel.m" \o "Matlab/Octave script for BER computation of BPSK in Rayleigh fading channel)



**Figure: BER plot of BPSK in Rayleigh fading channel**

When compared to the AWGN case, around **25dB degradation due to the multipath** channel (at the http://www.dsplog.com/cgi-bin/mimetex.cgi?10%5e%7b-4%7dpoint). This is both good and bad: bad because we need to spend so much energy to get a reliable wireless link up (in this era of global warming), and good because we signal processing engineers are trying to figure out ways for improving the performance.

**Reference**

[[DIGITAL COMMUNICATION: PROAKIS] Digital Communications by John Proakis](http://www.amazon.com/gp/redirect.html?ie=UTF8&location=http%3A%2F%2Fwww.amazon.com%2FDigital-Communications-John-Proakis%2Fdp%2F0072321113&tag=dl04-20&linkCode=ur2&camp=1789&creative=9325)